On the first day of March, a bank loans a man £2500 at a fixed rate of interest of 1·5% per month. This interest is added on the last day of each month and is calculated on the amount due on the first day of the month. He agrees to make repayments on the first day of each subsequent month. Each repayment is £300 except for the smaller final amount which will pay off the loan.

(a) The amount that he owes at the start of each month is taken to be the amount still owing just after the monthly repayment has been made.

Let u_n and u_{n+1} represent the amounts that he owes at the starts of two successive months. Write down a recurrence relation involving u_{n+1} and u_n. **2**

(b) Find the date and the amount of the final payment. **4**

a) $U_{n+1} = 1 \cdot 015 U_n - 300$ ✓ ✓

$\left(100\% + 1 \cdot 5\% = 101 \cdot 5\% = 1 \cdot 015\right)$

b)
1st Mar $U_0 = 2500$

1st Apr $U_1 = 1 \cdot 015 \times 2500 - 300 = £2237 \cdot 50$

1st May $U_2 = 1 \cdot 015 \times 2237 \cdot 50 - 300 = £1971 \cdot 06$ ✓

1st Jun $U_3 = £1700 \cdot 63$

1st Jul $U_4 = £1426 \cdot 14$

1st Aug $U_5 = £1147 \cdot 53$

1st Sept $U_6 = £864 \cdot 74$ ✓

1st Oct $U_7 = £577 \cdot 71$

1st Nov $U_8 = £286 \cdot 38$ ✓

1st Dec $U_9 = -£9 \cdot 32$ (ie, last repayment is too much!)

Final repayment $= 300 - 9 \cdot 32$
$= £290 \cdot 68$ ✓

on 1st December

Q2.

A recurrence relation is defined by $u_{n+1} = pu_n + q$, where $-1 < p < 1$ and $u_0 = 12$.

(*a*) If $u_1 = 15$ and $u_2 = 16$, find the values of p and q. **2**

(*b*) Find the limit of this recurrence relation as $n \to \infty$. **2**

a) $U_{n+1} = pU_n + q$ $\qquad$ $U_{n+1} = pU_n + q$

$(U_1 = pU_0 + q)$ $\qquad$ $(U_2 = pU_1 + q)$

$15 = 12p + q$ ① $\qquad$ $16 = 15p + q$ ②

$$\begin{aligned} 15 &= 12p + q \\ 16 &= 15p + q \\ \hline 1 &= 3p \end{aligned}$$

$\ominus$

$p = \frac{1}{3}$

find q

$15 = 12p + q$

$15 = 12 \times \frac{1}{3} + q$

$15 = 4 + q$

$q = 11$

b) $L = \frac{b}{1-a}$

$= \frac{11}{1-\frac{1}{3}}$

$= \frac{11}{\frac{2}{3}}$

$= 11 \div \frac{2}{3} = 11 \times \frac{3}{2} = \frac{33}{2}$ or $16 \cdot 5$

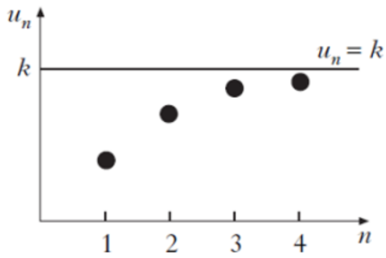
A sequence is defined by the recurrence relation

$$u_{n+1} = \frac{1}{4}u_n + 16, \; u_0 = 0.$$

(*a*) Calculate the values of u_1, u_2 and u_3. **3**

Four terms of this sequence, u_1, u_2, u_3 and u_4 are plotted as shown in the graph.

As $n \to \infty$, the points on the graph approach the line $u_n = k$, where k is the limit of this sequence.



(*b*) (i) Give a reason why this sequence has a limit.

(ii) Find the exact value of k. **3**

a) $U_{n+1} = \frac{1}{4}U_n + 16$

$U_1 = \frac{1}{4} \times 0 + 16$

$= 16$ ✓

$U_2 = \frac{1}{4} \times 16 + 16$

$= 4 + 16$

$= 20$ ✓

$U_3 = \frac{1}{4} \times 20 + 16$

$= 5 + 16$

$= 21$ ✓

b) i) Limit exists because $-1 < \frac{1}{4} < 1$

(ie, $\frac{1}{4}$ lies between -1 and 1) ✓

ii) $k = \frac{b}{1-a}$

$= \frac{16}{1 - \frac{1}{4}}$ ✓

$= \frac{16}{\frac{3}{4}} = 16 \div \frac{3}{4} = 16 \times \frac{4}{3} = \frac{64}{3}$ or $21\frac{1}{3}$ ✓

Q4.

Two sequences are generated by the recurrence relations $u_{n+1} = au_n + 10$ and $v_{n+1} = a^2v_n + 16$.

The two sequences approach the same limit as $n \to \infty$.

Determine the value of a and evaluate the limit. **5**

$U_{n+1} = aU_n + 10$ $\quad\quad$ $V_{n+1} = a^2 V_n + 16$

$L = \frac{b}{1-a}$ $\quad\quad$ $L = \frac{b}{1-a}$

$= \frac{10}{1-a}$ ✓ $\quad\quad$ $= \frac{16}{1-a^2}$

$$\frac{10}{1-a} = \frac{16}{1-a^2}$$ ✓

$$10(1-a^2) = 16(1-a)$$

$$10 - 10a^2 = 16 - 16a$$

$$10a^2 - 16a + 6 = 0$$ ✓

$$5a^2 - 8a + 3 = 0$$

$$(5a-3)(a-1) = 0$$

$5a - 3 = 0$ $\quad\quad$ $a - 1 = 0$

$5a = 3$ $\quad\quad$ $\cancel{a = 1}$

$a = \frac{3}{5}$

So $a = \frac{3}{5}$ since $-1 < a < 1$. ✓

$$L = \frac{b}{1-a} = \frac{10}{1-\frac{3}{5}} = \frac{10}{\frac{2}{5}} = 10 \div \frac{2}{5} = 10 \times \frac{5}{2} = 25$$ ✓