Higher Homework - The circle

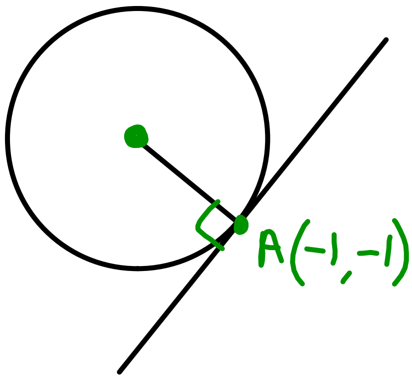
1\. a) This diagram shows a circle, centre P, with equation
$x^2 + y^2 + 6x + 4y + 8 = 0$.
Find the equation of the tangent at the point A(-1,-1) on the circle. 4

$x^2 + y^2 + 6x + 4y + 8 = 0$

$2g = 6 \quad 2f = 4$

$g = 3 \quad f = 2$

Centre $(-3, -2)$ ✓



$(-1, -1)$
$a \quad b$

$$M_{RAD} = \frac{-2+1}{-3+1} = \frac{-1}{-2} = \frac{1}{2} \checkmark$$

So $M_{tgt} = -2$ ✓

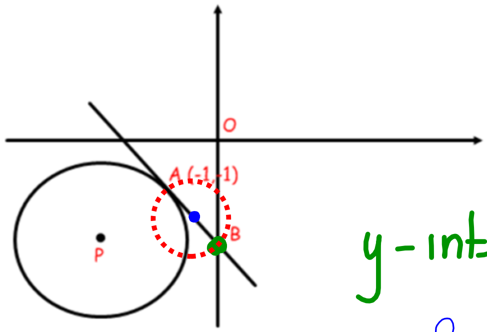
$y - b = m(x - a)$

$y + 1 = -2(x + 1)$

$y + 1 = -2x - 2$

$2x + y + 3 = 0$ ✓

b) The tangent crosses the y-axis at B.
Find the equation of the circle with AB as diameter. 3



y-intercept (Point B)

$$2x + y + 3 = 0$$

$$y = -2x - 3$$ So B is $(0,-3)$

Centre of Circle is mid-pt of AB $= \left(\frac{-1+0}{2}, \frac{-1-3}{2}\right)$

$$= \left(\underbrace{-\tfrac{1}{2}}_{a}, \underbrace{-2}_{b}\right)$$

$$(x-a)^2 + (y-b)^2 = r^2$$

$$\left(x+\tfrac{1}{2}\right)^2 + (y+2)^2 = r^2$$

find r^2, use $(-1,-1)$ or $(\underbrace{0}_{x}, \underbrace{-3}_{y})$ (Both points lie on circumference)

$$\left(0+\tfrac{1}{2}\right)^2 + (-3+2)^2 = r^2$$

$$\frac{1}{4} + 1 = r^2$$

$$r^2 = \frac{5}{4}$$

$$\left(x+\tfrac{1}{2}\right)^2 + (y+2)^2 = \frac{5}{4}$$

2. For what range of values of k does the equation
$x^2 + y^2 + 4kx - 2ky - k + 4 = 0$ represent a circle. 5

This suggests you require an inequality.

If a circle then radius > 0

So $g^2 + f^2 - c > 0$ ✓

$2g = 4k \quad 2f = -2k$

$g = 2k \quad f = -k$

$(2k)^2 + (-k)^2 - (4-k) > 0$ ✓

$c = -k + 4$
$= 4 - k$

$4k^2 + k^2 - 4 + k > 0$

$5k^2 + k - 4 > 0$ ✓ (ie, $y > 0$)

(this is a quadratic inequality)

Roots ($y = 0$)

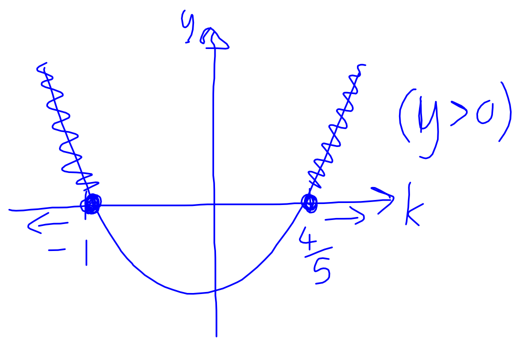
$5k^2 + k - 4 = 0$

$(5k - 4)(k + 1) = 0$

$k = \frac{4}{5} \quad k = -1$

So $k < -1$

and $k > \frac{4}{5}$



3. Circle P has equation $x^2 + y^2 - 8x - 10y + 9 = 0$. Circle Q has centre $(-2,-1)$ and radius $2\sqrt{2}$.

a) (i) Show that the radius of the circle P is $4\sqrt{2}$.

(ii) Hence show that the circles P and Q touch. 4

a) i) $x^2+y^2-8x-10y+9=0$

$2g=-8 \quad 2f=-10 \quad c=9$

$g=-4 \quad f=-5$

$r=\sqrt{g^2+f^2-c}$

$=\sqrt{(-4)^2+(-5)^2-9}$

$=\sqrt{16+25-9}$

$=\sqrt{32}$

$=\sqrt{16}\sqrt{2}$

$=4\sqrt{2}$ ✓

ii) Centre $(4,5)$ ✓

Distance between centres

$=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$=\sqrt{(4+2)^2+(5+1)^2}$

$=\sqrt{6^2+6^2}$

$=\sqrt{36+36}$

$=\sqrt{72}$

$=\sqrt{36}\sqrt{2}$

$=6\sqrt{2}$ ✓

radius(P) + radius(Q)

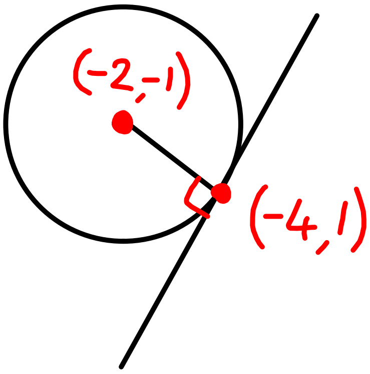
$=4\sqrt{2}+2\sqrt{2}$

$=6\sqrt{2}$ ✓ (and conclusion)

Since the distance between the centres is equal to the radius of each circle added together then the circles touch.

b) Find the equation of the tangent to circle Q at the point (-4, 1)

3



$$m_{RAD} = \frac{1+1}{-4+2}$$

$$= \frac{2}{-2}$$

$$= -1$$ ✓

So $m_{tgt} = 1$ ✓ $(\underset{a}{-4}, \underset{b}{1})$

$$y - b = m(x - a)$$

$$y - 1 = 1(x + 4)$$

$$y - 1 = x + 4$$

$$x - y + 5 = 0$$ ✓

4. The Line y + 2x = k, k is greater than 0, is a tangent to the circle $x^2 + y^2 - 2x - 4 = 0$.

a) Find the value of k. 7

$$y + 2x = k$$
$$y = k - 2x$$

$$x^2 + y^2 - 2x - 4 = 0$$
$$x^2 + (k-2x)(k-2x) - 2x - 4 = 0 \checkmark$$
$$x^2 + k^2 - 2kx - 2kx + 4x^2 - 2x - 4 = 0$$
$$5x^2 - 4kx - 2x + k^2 - 4 = 0$$
$$\underbrace{5}_{a}x^2 - x\underbrace{(4k+2)}_{b} + \underbrace{k^2-4}_{c} = 0 \checkmark$$

Since line is a tangent $b^2 - 4ac = 0$

$a = 5$

$b = -(4k+2)$

$c = k^2 - 4$

$$(4k+2)(4k+2) - 4 \times 5 \times (k^2-4) = 0 \checkmark$$
$$16k^2 + 8k + 8k + 4 - 20(k^2-4) = 0$$
$$16k^2 + 16k + 4 - 20k^2 + 80 = 0$$
$$-4k^2 + 16k + 84 = 0$$
$$4k^2 - 16k - 84 = 0$$
$$k^2 - 4k - 21 = 0 \checkmark$$
$$(k - 7)(k + 3) = 0$$
$$k - 7 = 0 \text{ or } k + 3 = 0$$
$$k = 7 \qquad k = -3$$

So $k = 7$ since $k > 0$. ✓

b) Deduce the coordinates of the point of contact. 2

from (a) $5x^2 - x(4k+2) + k^2 - 4 = 0$

$k=7$ $5x^2 - x(4\times7+2) + 7^2 - 4 = 0$

$$5x^2 - 30x + 45 = 0$$

$$x^2 - 6x + 9 = 0$$

$$(x-3)(x-3) = 0$$

$x - 3 = 0$ (twice)

$$x = 3$$

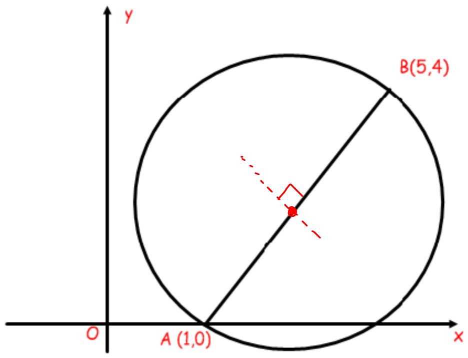
find y $y = k - 2x$

$$= 7 - 2\times3$$

$$= 1$$

$(3, 1)$

5.a) A chord joins the points A(1,0) and B(5,4) on the circle as shown in the diagram.
Show that the equation of the perpendicular bisector of chord AB is
x + y =5



Don't assume AB is a diameter!
(Never assume anything!)

$$\text{Mid-pt } AB = \left(\frac{1+5}{2}, \frac{0+4}{2}\right)$$
$$= (3, 2)$$

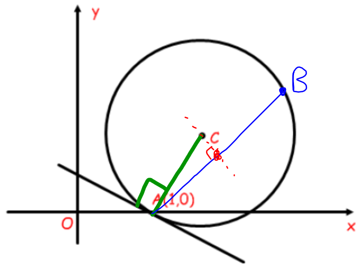
a b

$$M_{AB} = \frac{4-0}{5-1}$$
$$= \frac{4}{4}$$
$$= 1$$

So $M_{perp} = -1$

$$y - b = m(x - a)$$
$$y - 2 = -1(x - 3)$$
$$y - 2 = -x + 3$$
$$x + y = 5$$

b) The point C is the centre of this circle. The tangent at the point A on the circle has equation $x + 3y = 1$
Find the equation of the radius CA. 4



c) (i) Determine the coordinates of the point C.
(ii) Find the equation of the circle.
4

b) $x + 3y = 1$

$3y = 1 - x$

$y = \frac{1}{3} - \frac{1}{3}x$ So $M_{tgt} = -\frac{1}{3}$ and $M_{RAD} = 3$ $(\underset{a}{1}, \underset{b}{0})$

$y - b = m(x - a)$

$y - 0 = 3(x - 1)$

$y = 3x - 3$

c) i) Centre is point of intersection between radius and perp bisector. Use simultaneous equations.

$y = 3x - 3$

$x + y = 5$

$x + 3x - 3 = 5$

$4x - 3 = 5$

$4x = 8$

$x = 2$

find y

$y = 3 \times 2 - 3$

$= 6 - 3$

$= 3$ Centre $(\overset{a}{2}, \overset{b}{3})$

c) ii) $(x - a)^2 + (y - b)^2 = r^2$

$(x - 2)^2 + (y - 3)^2 = r^2$ find r^2 use $(\underset{x}{1}, \underset{y}{0})$

$(1 - 2)^2 + (0 - 3)^2 = r^2$

$(-1)^2 + (-3)^2 = r^2$

$1 + 9 = r^2$

$r^2 = 10$ So $(x - 2)^2 + (y - 3)^2 = 10$

6. Two congruent circles, with centres A and B touch at P. Relative to suitable axes, their equations are

$$x^2 + y^2 + 6x + 4y - 12 = 0 \text{ and}$$
$$x^2 + y^2 - 6x - 12y + 20 = 0.$$

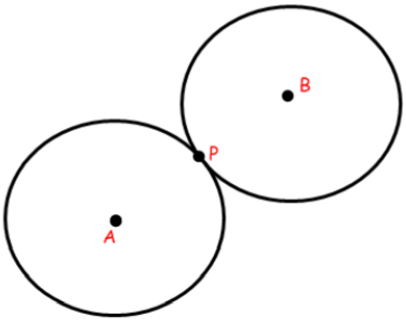
a) Find the coordinates of P. 3

b) Find the length of AB. 1

a) Circle A $\quad 2g = 6 \quad 2f = 4 \quad c = -12$

$g = 3 \quad f = 2$

Centre $(-3, -2)$ ✓



Circle B $\quad 2g = -6 \quad 2f = -12 \quad c = 20$

$g = -3 \quad f = -6$

Centre $(3, 6)$ ✓

P is the mid-pt of AB $= \left(\frac{-3+3}{2}, \frac{-2+6}{2}\right) = (0, 2)$ ✓

b) $d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(3+3)^2 + (6+2)^2}$ ✓

$= \sqrt{6^2 + 8^2}$

$= \sqrt{100}$

$= 10$ units ✓

or find radius of one circle using $r = \sqrt{g^2 + f^2 - c}$ and double.